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HYPERSURFACES SATISFYING PSEUDO-SYMMETRY CONDITIONS FOR THEIR WEYL CONFORMAL CURVATURE TENSOR

J. DEPREZ - R. DESZCZ - L. VERSTRAELEN* - S. YAPRAK
(* socio corrispondente)

Résumé

Nous étudions les hypersurfaces d'un espace Euclidien E^{n+1} qui satisfont la condition $C \cdot C = f\,Q(g, C)$. Nous démontrons que toutes les hypersurfaces qui ont jusqu'à trois courbures principales dont les multiplicitiés sont 1, 1 et $n-2$ satisfont cette condition. Nous étudions aussi la condition $C \cdot R = f\,Q(g, R)$. Une hypersurface M satisfait cette condition si et seulement si M a jusqu'à deux courbures principales distinctes ou le numéro du type de M est deux.

Abstract

We study the hypersurfaces of Euclidean space E^{n+1} satisfying the condition $C \cdot C = f\,Q(g, C)$. We prove that all hypersurfaces with at most three distinct principal curvatures with multiplicities, 1, 1 and $n-2$ satisfy this condition. We also examine the condition $C \cdot R = f\,Q(g, R)$. We show that a hypersurface M satisfy this condition if and only if M has at most two distinct principal curvatures or the type number of M is two.

1. Introduction.

In this paper we investigate the hypersurfaces of a Euclidean space satisfying one of the conditions:

$$(*) \qquad C \cdot C = f\,Q(g, C)$$

or

$$(**) \qquad C \cdot R = f\,Q(g, R),$$

where C denotes the Weyl conformal curvature tensor and R the Riemann-Christoffel curvature tensor of the hypersurface. The condition (*) arose during the study of 4-dimensional warped product manifolds (see Theorem 2 of [D1]). Some further results on manifolds satisfying ($*$) are given in [DD] and [DVY]. Many related curvature conditions have been studied by several autors. Among them, *semi-symmetric spaces*, i.e. Riemannian manifolds for which $R \cdot R = 0$, are a natural generalization of the locally symmetric manifolds. In [N] K. Nomizu studied semi-symmetric hypersurfaces of Euclidean spaces and P. J. Ryan examined semi-symmetric hypersurfaces of space forms [R1]. In [S1] and [S2] Z. I. Szabó obtained local and global classifications of the semi-symmetric Riemannian manifolds. D. E. Blair, P. Verheyen and L. Verstraelen studied the hypersurfaces of Euclidean space satisfying the conditions $R \cdot C = 0$ or $C \cdot R = 0$, [BVV].

The *pseudo-symmetric manifold*, i.e. the Riemannian manifolds for which $R \cdot R = f\,Q(g, R)$, form a natural extention of the semi-symmetric Riemannian manifolds. Every semi-symmetric manifold is a pseudo-symmetric manifold. The converse statement is not true (e.g. see [DDV2]). For a recent survey of pseudo-symmetric type curvature conditions on Riemannian or semi-Riemannian manifolds see [D2].

A manifold M is called (locally) conformally flat if M is (locally) conformally equivalent to E^N. It is well-known that M

is *conformally flat* if and only if $C = 0$ for $\dim M \geq 4$. We recall that every surface is conformally flat and C vanishes identically on every 3-dimensional Riemannian manifold. Therefore in the present paper we always assume that the dimension of the hypersurfaces will be ≥ 4. A hypersurface M of E^{N+1} is said to be *quasi-umbilical* if M has a principal curvature with multiplicity $\geq N-1$. E. Cartan proved that a hypersurfaces M of E^{N+1}, $N \geq 4$, is quasi-umbilical if and only if it is conformally flat.

In the present paper we obtain an algebraic classification of the hypersurfaces satisfying one of the conditions $(*)$ or $(**)$. We show that there are many non conformally flat hypersurfaces satisfying $C \cdot C = f\,Q(g, C)$. Namely, all hypersurfaces with three distinct principal curvatures, with multiplicities 1,1 and $N-2$ satisfy this condition. We remark that there are many examples of hypersurfaces satisfying this condition (see [P] and [CR] for more detailed information). We also prove that a hypersurface M of E^{N+1} satisfies $(**)$ if and only if M has at most two distinct principal curvatures or the type number of M is two (i.e. $R \cdot R = 0$). This, in view of Theorem 1 of [DDV1], implies that every hypersurface of E^{N+1} satisfying $(**)$ is pseudo-symmetric. More precisely, we prove the following theorems.

THEOREM 1. *Let M be a hypersurface of E^{N+1}, $N \geq 4$. If M satisfies $(*)$ then at every point $p \in U_C M$ has at most three distinct principal curvatures. Moreover, if $(*)$ is satisfied at $p \in U_C$ at which M has exactly three distinct principal curvatures then their multiplicities are the following: 1,1 and $N-2$. Conversely, if M has at a point $p \in U_C$ at most two distinct principal curvatures or M has at $p \in U_C$ three distinct principal curvatures with multiplicities 1,1 and $N-2$ then $(*)$ holds at p.*

Remark. A hypersurface M, $dim M = N$, having a principal curvature with multiplicity $\geq N-2$ is called *2-quasi-umbilical* [DMV].

THEOREM 2. *Let M be a hypersurface of E^{n+1}, $N \geq 4$. Then M satisfies $C \cdot R = fQ(g, R)$ if and only if at every point $p \in U_C M$ has at most two distinct principal curvatures or the type number of M (i.e. the rank of A_p) is two at p.*

In the above theorems by U_C we denote the subset of M consisting of all points of M at which the Weyl tensor of M is nonzero.

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2. Preliminaries.

Let M be a hypersurface of an $(N+1)$-dimensional Euclidean space E^{N+1} and let ξ be a local normal section on M. The formulas of Gauss and Weingarten are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi, \ \tilde{\nabla}_X \xi = -AX,$$

for vector fields X, Y, Z which are tangent to M; here $\tilde{\nabla}$ is the Euclidean connection on E^{N+1} and ∇ is the Levi-Civita connection on M, and the second fundamental tensor A is related to the second fundamental form h by $h(X, Y) = g(AX, Y)$, where g is the Riemannian metric on M. Let $X \wedge Y$ denote the endomorphism defined by $X \wedge Y : Z \mapsto g(Z, Y)X - g(Z, X)Y$.

Then the curvature operator $\tilde{R}$ of M is given by the equation of Gauss:

$$\tilde{R}(X,Y) = AX \wedge AY. \tag{2.1}$$

Since A is symmetric, there exists an orthonormal frame $e_1, e_2, \ldots, e_N$ consisting of eigenvectors, i.e. such that

$$Ae_i = \lambda_i e_i, \tag{2.2}$$

where $i \in \{1, 2, \ldots, N\}$, and $\lambda_1, \ldots, \lambda_N$ are the principal curvatures of M. The *Weyl conformal operator* $\tilde{C}$ of M is defined by

$$\tilde{C}(X,Y) = \tilde{R}(X,Y) - \frac{1}{(N-2)}(\tilde{S}X \wedge Y + X \wedge \tilde{S}Y) + \\ + \frac{\tau}{(N-1)(N-2)} X \wedge Y,$$

where $\tau = tr\tilde{S}$ is the scalar curvature of M and $\tilde{S}$ is the (1,1)-tensor related to the Ricci tensor S of M by $g(\tilde{S}X, Y) = S(X,Y)$. The Weyl conformal curvature tensor C of M is defined by

$$C(X_1, \ldots, X_4) = g(\tilde{C}(X_1, X_2)X_3, X_4),$$

for any vector fields $X_1, \ldots, X_4$ tangent to M.

By (2.2), the equation of Gauss implies that

$$\tilde{R}(e_i, e_j) = \lambda_i \lambda_j e_i \wedge e_j,$$

and consequently

$$\tilde{S}(e_i) = \mu_i e_i, \ \tilde{C}(e_i, e_j) = a_{ij} e_i \wedge e_j, \tag{2.3}$$

where

$$\mu_i = \lambda_i (trA - \lambda_i),$$

$$(2.4) \qquad a_{ij} = \lambda_i \lambda_j - \frac{1}{(N-2)}(\mu_i + \mu_j) + \frac{(trA)^2 - tr(A^2)}{(N-1)(N-2)},$$

for all i, j and $k \in \{1, \ldots, N\}$.

Let $\bar{\lambda}_1, \ldots, \bar{\lambda}_r$ be the mutually distinct eigenvalues of A with multiplicities $s_1, \ldots, s_r$, respectively. Denote by V_α the space of eigenvectors corresponding to the eigenvalue $\bar{\lambda}_\alpha$, $\alpha \in \{1, \ldots, r\}$. When $e_i, e_k \in V_\alpha$ and $e_j, e_l \in V_\beta$ then $a_{ij} = a_{kl}$ and $\mu_i = \mu_k$, for all $i, j, k, l \in \{1, \ldots, N\}$ and $\alpha, \beta \in \{1, \ldots, r\}$. We define the numbers $\bar{\mu}_\alpha = \mu_i$ and $\bar{a}_{\alpha\beta} = a_{ij}$, where $e_i \in V_\alpha e_j \in V_\beta$, and $i, j \in \{1, \ldots, N\}$ and $\alpha, \beta \in \{1, \ldots, r\}$.

Concerning the notations, by $C \cdot C = fQ(g, C)$, for example, we mean that $\tilde{C}(X, Y) \cdot C = -f(X \wedge Y) \cdot C$ for all vector fields X and Y on M, where $\tilde{C}(X, Y)$ and $X \wedge Y$ acts as derivations on the algebra of tensor fields on M, i.e.

$$(C \cdot C)(Z, U, V, W; X, Y) = (\tilde{C}(X, Y) \cdot C)(Z, U, V, W)$$

$$-C(\tilde{C}(X, Y)Z, U, V, W) - C(Z, \tilde{C}(X, Y)U, V, W)$$

$$-C(Z, U, \tilde{C}(X, Y)V, W) - C(Z, U, V, \tilde{C}(X, Y)W)$$

for X, Y, Z, U, V, W tangent to M.

For a $(0, s)$-tensor T on M a $(0, s+2)$-tensor $Q(g, T)$ is defined by

$$Q(g, T)(X_1, \ldots, X_s; Y, Z) = -((Y \wedge Z) \cdot T)(X_1, X_2, \ldots, X_s)$$

(see, e.g. [T]). We say that a Riemannian manifold (M, g) satisfies $C \cdot T = fQ(g, T)$, if there exists a function $f : M \to R$ such that

$$(\tilde{C}(Y, Z) \cdot T)(X_1, \ldots, X_s)(p) =$$

$$f(p)Q(g, T)(X_1, \ldots, X_s; Y, Z)(p)$$

for each p in M and all $X_1, \ldots, X_s, Y, Z$ tangent to M at p.

3. Proof of Theorem 1.

Let M be a hypersurface isometrically immersed of E^{N+1} and p a point in M. Suppose $\{e_1, \ldots, e_N\}$ is an orthonormal frame of T_pM satisfying (2.2). Using (2.3) we obtain that

$$(\tilde{C}(e_i, e_j) \cdot C)(e_k, e_l, e_m, e_n) - f(p)Q(g, C)(e_k, e_l, e_m, e_n; e_i, e_j)$$

$$= -(f(p) + a_{ij})\{\delta_{jk}a_{il}(\delta_{in}\delta_{lm} - \delta_{im}\delta_{ln}) - \delta_{ik}a_{jl}(\delta_{jn}\delta_{lm} - \delta_{jm}\delta_{ln})$$

$$+ \delta_{jl}a_{ik}(\delta_{im}\delta_{kn} - \delta_{in}\delta_{km}) - \delta_{ik}a_{jl}(\delta_{jn}\delta_{lm} - \delta_{jn}\delta_{km})$$

$$+ \delta_{jm}a_{kl}(\delta_{il}\delta_{kn} - \delta_{ik}\delta_{ln}) - \delta_{im}a_{kl}(\delta_{jl}\delta_{kn} - \delta_{jk}\delta_{ln})$$

$$+ \delta_{jn}a_{kl}(\delta_{ik}\delta_{lm} - \delta_{il}\delta_{km}) - \delta_{in}a_{kl}(\delta_{jk}\delta_{lm} - \delta_{jl}\delta_{km})\},$$

for all $i, j, k, l, m, n \in \{1, \ldots, N\}$. From this equation it is clear that $C \cdot C = fQ(g, C)$ at p if and only if

$$(\tilde{C}(e_i, e_j) \cdot C)(e_i, e_k, e_j, e_k) = f(p)Q(g, C)(e_i, e_k, e_j, e_k; e_i, e_j),$$

for all mutually distinct $i, j, k \in \{1, 2, \ldots, N\}$, which is equivalent to the following:

$$(f(p) + a_{ij})(a_{ik} - a_{jk}) = 0,$$

for all mutually distinct $i, j, k \in \{1, 2, \ldots, N\}$, which is equivalent to the following:

$$(f(p) + a_{ij})(a_{ik} - a_{jk}) = 0,$$

for all mutually distinct $i, j, k \in \{1, \ldots, N\}$, i.e.

$$(3.1) \quad (f(p) + a_{ij})(trA - \lambda_i - \lambda_j - (N-2)\lambda_k)(\lambda_i - \lambda_j) = 0,$$

for all mutually distinct i, j and $k \in \{1, \ldots, N\}$. Now we will consider several cases separately.

(i) If $r = 1$ then clearly C vanishes at p. So $(*)$ is trivially satisfied.

(ii) If $r = 2$ then we have $\bar{\lambda}_\alpha \neq \bar{\lambda}_\beta$ with multiplicities s_α and s_β, respectively.

If $s_\alpha = 1$ or $s_\beta = 1$ then by [CV], $\bar{a}_{\alpha\beta} = 0$ which means that C vanishes at p. Thus we may suppose that $s_\alpha \geq 2$ and $s_\beta \geq 2$. Then taking e.g. $\lambda_i = \bar{\lambda}_\alpha$, $\lambda_j = \bar{\lambda}_\beta$ and $\lambda_k = \bar{\lambda}_\alpha$ we can easily see that

$$tr A - \lambda_i - \lambda_j - (N-2)\lambda_k = (1 - s_\beta)(\bar{\lambda}_\alpha - \bar{\lambda}_\beta)$$

which does not vanish. Therefore (3.1) shows that $C \cdot C = f Q(g, C)$ at p for

$$f(p) = -\bar{a}_{\alpha\beta} = \frac{(1 - s_\alpha)(1 - s_\beta)}{(N-1)(N-2)}(\bar{\lambda}_\alpha - \bar{\lambda}_\beta)^2.$$

(iii) If $r = 3$, we can choose mutually distinct indices $\alpha, \beta, \gamma \in \{1, 2, 3\}$.

Assume that M satisfies $C \cdot C = f Q(g, C)$ at p. Then (3.1) implies that

$$(f(p) + \bar{a}_{\alpha\beta})(tr A - \bar{\lambda}_\alpha - \bar{\lambda}_\beta - (N-2)\bar{\lambda}_\gamma) = 0, \tag{3.2}$$

$$(f(p) + \bar{a}_{\beta\gamma})(tr A - \bar{\lambda}_\beta - \bar{\lambda}_\gamma - (N-2)\bar{\lambda}_\alpha) = 0, \tag{3.3}$$

$$(f(p) + \bar{a}_{\alpha\gamma})(tr A - \lambda_\alpha - \bar{\lambda}_\gamma - (N-2)\bar{\lambda}_\beta) = 0. \tag{3.4}$$

Now if

$$tr A - \bar{\lambda}_\beta - \bar{\lambda}_\gamma - (N-2)\bar{\lambda}_\alpha = 0$$

and

$$tr A - \bar{\lambda}_\alpha - \bar{\lambda}_\gamma - (N-2)\bar{\lambda}_\beta = 0,$$

subtraction yields

$$(N-3)(\bar{\lambda}_\alpha - \bar{\lambda}_\beta) = 0,$$

which is a contradiction since $N > 3$ and $\bar{\lambda}_\alpha \neq \bar{\lambda}_\beta$. Now let us assume that

$$trA - \lambda_\beta - \bar{\lambda}_\gamma - (N-2)\ddot{\lambda}_\alpha \neq 0, \tag{3.5}$$

$$trA - \bar{\lambda}_\alpha - \bar{\lambda}_\gamma - (N-2)\bar{\lambda}_\beta \neq 0. \tag{3.6}$$

Then (3.3) and (3.4) imply that

$$-f(p) = \bar{a}_{\alpha\gamma} = \bar{a}_{\beta\gamma}, \tag{3.7}$$

which gives

$$trA = \bar{\lambda}_\alpha + \bar{\lambda}_\beta + (N-2)\bar{\lambda}_\gamma. \tag{3.8}$$

We note that $f(p) + \bar{a}_{\alpha\beta}$ is nonzero. In fact, if we had $f(p) = -\bar{a}_{\alpha\beta}$ then by (3.7) we would obtain $trA = \bar{\lambda}_\beta + \bar{\lambda}_\gamma + (N-2)\bar{\lambda}_\alpha$ which contradicts to (3.5).

Now suppose that $s_\alpha > 1$. Putting

$$\lambda_i = \bar{\lambda}_\alpha,\ \lambda_j = \bar{\lambda}_\beta,\ \lambda_k = \bar{\lambda}_\gamma,$$

$$\lambda_i = \bar{\lambda}_\alpha,\ \lambda_j = \lambda_\beta,\ \lambda_k = \bar{\lambda}_\alpha,$$

in (3.1), we get

$$(f(p) + \bar{a}_{\alpha\beta})(trA - \bar{\lambda}_\alpha - \bar{\lambda}_\beta - (N-2)\bar{\lambda}_\gamma) = 0,$$

$$(f(p) + \bar{a}_{\alpha\beta})(trA - \bar{\lambda}_\alpha - \bar{\lambda}_\beta - (N-2)\bar{\lambda}_\alpha) = 0.$$

Subtraction yields

$$(f(p) + \bar{a}_{\alpha\beta})(N-2)(\bar{\lambda}_\alpha - \bar{\lambda}_\gamma) = 0,$$

which implies that $f(p) = -\bar{a}_{\alpha\beta}$. But as we have pointed out earlier this contradicts to (3.5). Therefore s_α must be 1. Similarly it can be verified that s_β also must be 1. Then $s_\gamma = N - 2$. Conversely one can verify easily that if $s_\alpha = 1$, $s_\beta = 1$ and $s_\gamma = N - 2$ then (3.1) is satisfied for

$$-f(p) = \bar{a}_{\alpha\gamma} (= \bar{a}_{\beta\gamma}).$$

(iv) Now suppose that $r \geq 4$ and choose mutually distinct indices $\alpha, \beta, \gamma, \delta \in \{1, \dots, r\}$ and assume that $C \cdot C = f Q(g, C)$ at p. From (3.1) it follows that

$$(f(p) + \bar{a}_{\alpha\beta})(trA - \bar{\lambda}_\alpha - \bar{\lambda}_\beta - (N-2)\bar{\lambda}_\gamma) = 0, \tag{3.9}$$

$$(f(p) + \bar{a}_{\alpha\beta})(trA - \bar{\lambda}_\alpha - \bar{\lambda}_\beta - (N-2)\bar{\lambda}_\delta) = 0, \tag{3.10}$$

$$(f(p) + \bar{a}_{\alpha\gamma})(trA - \bar{\lambda}_\alpha - \bar{\lambda}_\gamma - (N-2)\bar{\lambda}_\beta) = 0, \tag{3.11}$$

$$(f(p) + \bar{a}_{\alpha\gamma})(trA - \bar{\lambda}_\alpha - \bar{\lambda}_\gamma - (N-2)\bar{\lambda}_\delta) = 0, \tag{3.12}$$

$$(f(p) + \bar{a}_{\alpha\delta})(trA - \bar{\lambda}_\alpha - \bar{\lambda}_\delta - (N-2)\bar{\lambda}_\beta) = 0, \tag{3.13}$$

$$(f(p) + \bar{a}_{\alpha\delta})(trA - \bar{\lambda}_\alpha - \bar{\lambda}_\delta - (N-2)\bar{\lambda}_\gamma) = 0, \tag{3.14}$$

$$(f(p) + \bar{a}_{\beta\gamma})(trA - \bar{\lambda}_\beta - \bar{\lambda}_\gamma - (N-2)\bar{\lambda}_\alpha) = 0, \tag{3.15}$$

$$(f(p) + \bar{a}_{\beta\gamma})(trA - \bar{\lambda}_\beta - \bar{\lambda}_\gamma - (N-2)\bar{\lambda}_\delta) = 0, \tag{3.16}$$

$$(f(p) + \bar{a}_{\beta\delta})(trA - \bar{\lambda}_\beta - \bar{\lambda}_\delta - (N-2)\bar{\lambda}_\alpha) = 0, \tag{3.17}$$

$$(f(p) + \bar{a}_{\beta\delta})(trA - \bar{\lambda}_\beta - \bar{\lambda}_\delta - (N-2)\bar{\lambda}_\gamma) = 0, \tag{3.18}$$

$$(3.19)\qquad (f(p)+\bar{a}_{\gamma\delta})(trA-\bar{\lambda}_\gamma-\bar{\lambda}_\delta-(N-2)\bar{\lambda}_\alpha)=0,$$

$$(3.20)\qquad (f(p)+\bar{a}_{\gamma\delta})(trA-\bar{\lambda}_\gamma-\bar{\lambda}_\delta-(N-2)\bar{\lambda}_\beta)=0,$$

If $f(p)\neq -\bar{a}_{\alpha\beta}$ then (3.9) and (3.10) implies that

$$trA-\bar{\lambda}_\alpha-\bar{\lambda}_\beta-(N-2)\bar{\lambda}_\gamma=0,$$

$$trA-\bar{\lambda}_\alpha-\bar{\lambda}_\beta-(N-2)\bar{\lambda}_\delta=0.$$

Subtraction yields $(N-2)(\bar{\lambda}_\gamma-\bar{\lambda}_\delta)=0$, which contradicts to the assumption on the eigenvalues. Thus $f(p)=-\bar{a}_{\alpha\beta}$. Similarly, if $f(p)\neq -\bar{a}_{\alpha\gamma}$ then (3.11) and (3.12) lead to a contradiction. So $f(p)=-\bar{a}_{\alpha\gamma}$. Furthermore, if $f(p)\neq -\bar{a}_{\alpha\beta}$ then by (3.13) and (3.14) we again obtain a contradiction. Thus $f(p)=-\bar{a}_{\alpha\delta}$. But these three conditions yield

$$trA-\bar{\lambda}_\beta-\bar{\lambda}_\gamma-(N-2)\bar{\lambda}_\alpha=0,$$

$$trA-\bar{\lambda}_\beta-\bar{\lambda}_\delta-(N-2)\bar{\lambda}_\alpha=0.$$

Subtraction gives $(\bar{\lambda}\gamma-\bar{\lambda}_\delta)=0$, which is a contradiction. Proceeding in the same way we reach a contradiction in each case. This completes the proof.

Before we conclude this section we present some examples of hypersurfaces satisfying the condition $C\cdot C=fQ(g,C)$.

EXAMPLE 1. Let M be a surface in E^3. We consider the hypercylinder $\tilde{M}=M\times E^k$ in E^{k+3}. Suppose that the principal curvatures of M are λ and μ. Obviously the principal curvatures of a hypercylinder $\tilde{M}$ are $(\lambda,\mu,0,\dots,0)$, (here 0 occurs k-times). Consequently, by Theorem 1, $\tilde{M}$ satisfy the condition $C\cdot C=fQ(g,C)$. In particular, for a surface M with two nonzero distinct principal curvatures, $\tilde{M}$ is a hypersurface with three distinct principal curvatures of multiplicities 1,1 and k.

EXAMPLE 2. Let M be a surface in E^{k+2} given by the immersion $f : M \to E^{k+2}$ and BM be the bundle of unit normals to M. The hypersurface $\tilde{M}$ defined by the map

$$\phi_t : BM \mapsto E^{k+2},\ \phi_t(x, \xi) = F(x, t\xi) = f(x) + t\xi$$

is called teh *tube of radius* t over M. It was proved in [CR] that if λ and μ are the principal curvatures of M then the principal curvatures of the tube over M are

$$\left(\frac{\lambda}{1-t\lambda}, \frac{\mu}{1-t\mu}, -\frac{1}{t}, \dots, -\frac{1}{t}\right),$$

here $-\dfrac{1}{t}$ occurs $(k-1)$-times. Hence the tubes over any surface M satisfy the condition $C \cdot C = fQ(g, C)$.

EXAMPLE 3. Let M be a surface in $E^{n+1} = E^2 \times E^{n-1}$ given by the parametrization $(u, v) \to (x_1(u, v), x_2(u, v), x_3(u, v), 0, \dots, 0)$. As described in [P], consider the rotations r_t of E^{n+1} that leave E^2 fixed. Let $\tilde{M}$ be the hypersurface of E^{n+1} generated by M under the rotations r_t. More precisely consider the hypersurfaces $\tilde{M}$ are given as follows:

$$\tilde{M} = \{(x_1(u, v), x_2(u, v), x_3(u, v)\cos\theta_3, x_3(u, v)\sin\theta_3\cos\theta_4,$$

$$x_3(u, v)\sin\theta_3\sin\theta_4, \dots, x_3(u, v)\sin\theta_3\sin\theta_4\dots\cos\theta_n,$$

$$x_3(u, v)\sin\theta_3\sin\theta_4\dots\sin\theta_n | u, v, \theta_3, \theta_4, \dots, \theta_n \in R\}$$

$$= \{(x_1(u, v), x_2(u, v), 0, \dots, 0)$$

$$+ x_3(u, v)\varphi | \varphi : S^{n-2} \to E^{n-1}, u, v \in R\}.$$

It is clear that $\tilde{M}$ has principal curvatures of the form $(\lambda, \mu, \gamma, \dots, \gamma)$, where γ occurs $(n-2)$-times. Thus $\tilde{M}$ satisfies the condition $C \cdot C = fQ(g, C)$.

4. Proof of Theorem 2.

Let p be a point in M and $\{e_1, e_2, \ldots, e_N\}$ a basis of T_pM satisfying (2.2).

Using the relations given in (2.3), we obtain

$$(\tilde{C}(e_i, e_j) \cdot R)(e_k, e_l, e_m, e_n) - f(p)Q(g, R)(e_k, e_l, e_m, e_n; e_i, e_j)$$

$$= -(f(p) + a_{ij})\{\delta_{jk}\lambda_i\lambda_l(\delta_{in}\delta_{lm} - \delta_{im}\delta_{ln}) - \delta_{ik}\lambda_j\lambda_l(\delta_{jn}\delta_{lm} - \delta_{jm}\delta_{ln})$$

$$+ \delta_{jl}\lambda_i\lambda_k(\delta_{im}\delta_{kn} - \delta_{in}\delta_{km}) - \delta_{il}\lambda_j\lambda_k(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})$$

$$+ \delta_{jm}\lambda_k\lambda_l(\delta_{il}\delta_{kn} - \delta_{ik}\delta_{ln}) - \delta_{im}\lambda_k\lambda_l(\delta_{jl}\delta_{kn} - \delta_{jk}\delta_{ln})$$

$$+ \delta_{jn}\lambda_k\lambda_l(\delta_{ik}\delta_{lm} - \delta_{il}\delta_{km}) - \delta_{in}\lambda_k\lambda_l(\delta_{jk}\delta_{lm} - \delta_{jl}\delta_{km})\},$$

for all $i, j, k, l, m, n \in \{1, \ldots, N\}$. From this expression we see that $C \cdot R = fQ(g, R)$ at a point p if and only if

$$(\tilde{C}(e_i, e_j) \cdot R)(e_i, e_k, e_j, e_k) = f(p)Q(g, R)(e_i, e_k, e_j, e_k; e_i, e_j)$$

for all mutually distinct $i, j, k \in \{1, \ldots, N\}$. This last equality is equivalent to the following:

$$(4.1) \qquad (f(p) + a_{ij})(\lambda_i - \lambda_j)\lambda_k = 0,$$

for all mutually distinct indices i, j and $k \in \{1, \ldots, N\}$. Let $\bar{\lambda}_1, \ldots, \bar{\lambda}_r$ be mutually distinct eigenvalues of A at p with respective multiplicities $s_1, \ldots, s_r$.

We will examine the condition (4.1) in several cases separately.

(i) If $r = 1$ then $(**)$ is clearly satisfied at p.

(ii) If $r = 2$ then we have $\bar{\lambda}_\alpha$ and $\bar{\lambda}_\beta$ such that $\bar{\lambda}_\alpha \neq \bar{\lambda}_\beta$. It follows from (4.1) that $C \cdot R = fQ(g, R)$ at p for

$$f(p) = -\bar{a}_{\alpha\beta} = \frac{(1 - s_\alpha)(1 - s_\beta)}{(N-)(N - 2)}(\bar{\lambda}_\alpha - \bar{\lambda}_\beta)^2.$$

(iii) Now suppose that $r = 3$ and choose mutually distinct indices $\alpha, \beta, \gamma \in \{1, 2, 3\}$. Suppose that M satisfies $C \cdot R = f Q(g, R)$ at p. Moreover, assume that $s_\alpha \geq 2$. Then from (4.1) we obtain the following equations

$$(f(p) + \bar{a}_{\alpha\beta})\bar{\lambda}_\alpha = 0, \tag{4.2}$$

$$(f(p) + \bar{a}_{\alpha\gamma})\bar{\lambda}_\alpha = 0, \tag{4.3}$$

$$(f(p) + \bar{a}_{\beta\gamma})\bar{\lambda}_\alpha = 0. \tag{4.4}$$

From (4.2) and (4.3) we have $\bar{\lambda}_\alpha(\bar{a}_{\alpha\beta} - \bar{a}_{\alpha\gamma}) = 0$, i.e.

$$\bar{\lambda}_\alpha(trA - \bar{\lambda}_\beta - \bar{\lambda}_\gamma - (N-2)\bar{\lambda}_\alpha) = 0. \tag{4.5}$$

Similarly from (4.3) and (4.4) it follows that

$$\bar{\lambda}_\alpha(trA - \bar{\lambda}_\alpha - \bar{\lambda}_\beta - (N-2)\bar{\lambda}_\gamma) = 0. \tag{4.6}$$

Now (4.5) and (4.6) imply that $\bar{\lambda}_\alpha = 0$. Since $\bar{\lambda}_\alpha$, $\bar{\lambda}_\beta$, $\bar{\lambda}_\gamma$ are mutually distinct s_β and s_γ must be one. Thus the rank of M is two. But for this case $(**)$ is trivially satisfied with $f(p) = -\bar{a}_{\alpha\beta} = -\bar{a}_{\alpha\gamma}$.

(iv) Finally, suppose that $r \geq 4$ and choose mutually distinct $\alpha, \beta, \gamma, \delta \in \{1, 2, \ldots, r\}$. Assume that M satisfies the condition $C \cdot R = f Q(g, R)$. Then (4.1) implies that

$$(f(p) + \bar{a}_{\alpha\beta})\bar{\lambda}_\gamma = 0, \tag{4.7}$$

$$(f(p) + \bar{a}_{\alpha\beta})\bar{\lambda}_\delta = 0, \tag{4.8}$$

$$(f(p) + \bar{a}_{\alpha\gamma})\bar{\lambda}_\beta = 0, \tag{4.9}$$

$$(4.10)\qquad (f(p)+\bar{a}_{\alpha\gamma})\bar{\lambda}_{\delta}=0,$$

$$(4.11)\qquad (f(p)+\bar{a}_{\alpha\delta})\bar{\lambda}_{\beta}=0,$$

$$(4.12)\qquad (f(p)+\bar{a}_{\alpha\delta})\bar{\lambda}_{\gamma}=0,$$

$$(4.13)\qquad (f(p)+\bar{a}_{\beta\gamma})\bar{\lambda}_{\beta}=0,$$

$$(4.14)\qquad (f(p)+\bar{a}_{\beta\gamma})\bar{\lambda}_{\delta}=0,$$

$$(4.15)\qquad (f(p)+\bar{a}_{\beta\gamma})\bar{\lambda}_{\alpha}=0,$$

$$(4.16)\qquad (f(p)+\bar{a}_{\beta\delta})\bar{\lambda}_{\gamma}=0,$$

$$(4.17)\qquad (f(p)+\bar{a}_{\gamma\delta})\bar{\lambda}_{\alpha}=0,$$

$$(4.18)\qquad (f(p)+\bar{a}_{\gamma\delta})\bar{\lambda}_{\beta}=0,$$

If $f(p)\neq -\bar{a}_{\alpha\beta}$ then (4.7) and (4.8) imply that $\bar{\lambda}_{\gamma}=\bar{\lambda}_{\delta}=0$, which is a contradiction; thus $f(p)=-\bar{a}_{\alpha\beta}$. If $f(p)\neq -\bar{a}_{\alpha\gamma}$ then (4.9) and (4.10) give a contradiction; thus $f(p)=-\bar{a}_{\alpha\gamma}$. Similarly, from (4.11) and (4.12) we obtain that $f(p)=-\bar{a}_{\alpha\delta}$. These last three conclusions imply that

$$tr A-\bar{\lambda}_{\beta}-\bar{\lambda}_{\gamma}-(N-2)\bar{\lambda}_{\alpha}=0,$$

$$tr A-\bar{\lambda}_{\delta}-\bar{\lambda}_{\gamma}-(N-2)\bar{\lambda}_{\alpha}=0.$$

Subtraction yields $\bar{\lambda}_{\beta}=\bar{\lambda}_{\delta}$, which is a contradiction. Proceeding in the same way, we obtain contradictions in all possible cases. This completes the proof.

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J. Deprez - L. Verstraelen
Departement Wiskunde, Katholieke Universiteit Leuven
Celestijnenlaan 200 B, B-3001 Leuven, Belgium

Ryszard Deszcs
Department of Mathematics
Agricultural University of Wroclaw
ul. Grunwaldzka 53, PL-50357 Wroclaw, Poland

Sahnur Yaprak
Department of Mathematics
University of Ankara
06100 Tandogan, Ankara, Turkey